

# On Some Weighted Average Values of $L$ -functions

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## Abstract

Let  $q \geq 2$  and  $N \geq 1$  be integers. W. Zhang (2008) has shown that for any fixed  $\varepsilon > 0$ , and  $q^\varepsilon \leq N \leq q^{1/2-\varepsilon}$ ,

$$\sum_{\chi \neq \chi_0} \left| \sum_{n=1}^N \chi(n) \right|^2 |L(1, \chi)|^2 = (1 + o(1)) \alpha_q q N,$$

where the sum is take over all nonprincipal characters  $\chi$  modulo  $q$ ,  $L(s, \chi)$  is the  $L$ -functions  $L(1, \chi)$  corresponding to  $\chi$  and  $\alpha_q = q^{o(1)}$  is some explicit function of  $q$ . Here we improve this result and show that the same asymptotic formula holds in the essentially full range  $q^\varepsilon \leq N \leq q^{1-\varepsilon}$ .

**Key Words:**  $L$ -function, character sum, average value.

**Mathematical Subject Classification:** 11M06

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# 1 Introduction

For integers  $q \geq 2$  and  $N \geq 1$  we consider the average value

$$S(q; N) = \sum_{\chi \neq \chi_0} \left| \sum_{n=1}^N \chi(n) \right|^2 |L(1, \chi)|^2$$

taken over all nonprincipal characters  $\chi$  modulo an integer  $q \geq 2$ , with  $L$ -functions  $L(1, \chi)$  corresponding to  $\chi$ , weighted by incomplete character sums.

W. Zhang [2] has given an asymptotic formula for  $S(q; N)$  that is non-trivial for  $q^\varepsilon \leq N \leq q^{1/2-\varepsilon}$  for any fixed  $\varepsilon > 0$  and sufficiently large  $q$ .

Here we improve the error term of that formula which makes it nontrivial in the range  $q^\varepsilon \leq N \leq q^{1-\varepsilon}$ .

More precisely, let

$$\alpha_q = (\beta_q + \gamma_q) \frac{\varphi(q)^2}{q^2},$$

where

$$\begin{aligned} \beta_q &= \frac{\pi^2}{6} \prod_{\substack{p|q \\ p \text{ prime}}} \left(1 - \frac{1}{p^2}\right), \\ \gamma_q &= \frac{\pi^2}{3\zeta(3)} \prod_{\substack{p|q \\ p \text{ prime}}} \left(1 - \frac{1}{p^2 + p + 1}\right) \sum_{\substack{m, n=1 \\ \gcd(nm(n+m), q)=1}} \frac{1}{nm(n+m)}, \end{aligned}$$

$\zeta(s)$  is the Riemann zeta-function and  $\varphi(q)$  denotes the Euler function.

It is shown in [2] that

$$S(q, N) = \alpha_q q N + O\left(\varphi(q) 2^{\omega(q)} (\log q)^2 + N^3 (\log q)^2\right), \quad (1)$$

where  $\omega(q)$  is the number of prime divisors of  $q$ .

Since

$$2^{\omega(q)} \leq \tau(q) = q^{o(1)} \quad \text{and} \quad \varphi(q) = q^{1+o(1)}, \quad (2)$$

where  $\tau(q)$  is the number of positive integer divisors of  $q$ , see [1, Theorems 317 and 328], we conclude that  $\alpha_q = q^{o(1)}$  and the error in (1) is of the shape  $O(q^{1+o(1)} + N^3 q^{o(1)})$ .

In particular, the asymptotic formula (1) is nontrivial if  $q^\varepsilon \leq N \leq q^{1/2-\varepsilon}$  for any fixed  $\varepsilon > 0$  and  $q$  is large enough.

Here we estimate a certain sum which arises in [2] in a more accurate way and essentially replace  $N^3$  in (1) with  $N^2 q^{o(1)}$  which makes it nontrivial in the range  $q^\varepsilon \leq N \leq q^{1-\varepsilon}$ .

## 2 Main Result

**Theorem.** *Let  $q > N \geq 1$  be integers. Then*

$$S(q, N) = \alpha_q q N + O\left(\varphi(q) 2^{\omega(q)} (\log q)^2 + N^2 q^{o(1)}\right),$$

as  $q \rightarrow \infty$ .

*Proof.* It has been shown in [2] that

$$S(q, N) = M_1 + M_2 + O\left(N^2 (\log q)^2\right),$$

where

$$M_1 = \varphi(q) \sum_{m,n=1}^N \sum_{\substack{u,v=1 \\ mu=nv}}^{q^2} \frac{1}{uv} \quad \text{and} \quad M_2 = \varphi(q) \sum_{m,n=1}^N \sum_{\substack{u,v=1 \\ mu \equiv nv \pmod{q} \\ mu \neq nv}}^{q^2} \frac{1}{uv}.$$

Furthermore, it is shown in [2] that

$$M_1 = \alpha_q q N + O\left(\varphi(q) 2^{\omega(q)} (\log q)^2\right).$$

Thus, it remains to show that

$$M_2 \leq N^2 q^{o(1)}. \tag{3}$$

Let

$$J = \lfloor 2 \log q \rfloor.$$

Then, changing the order of summation, we obtain

$$\begin{aligned}
M_2 &= \varphi(q) \sum_{u,v=1}^{q^2} \frac{1}{uv} \sum_{\substack{m,n=1 \\ mu \equiv nv \pmod{q} \\ mu \neq nv}}^N 1 \\
&\leq \varphi(q) \sum_{i,j=0}^J \sum_{e^i \leq u < e^{i+1}} \frac{1}{u} \sum_{e^j \leq v < e^{j+1}} \frac{1}{v} \sum_{\substack{m,n=1 \\ mu \equiv nv \pmod{q} \\ mu \neq nv}}^N 1 \\
&\leq 2\varphi(q) \sum_{0 \leq i \leq j \leq J} \sum_{e^i \leq u < e^{i+1}} \frac{1}{u} \sum_{e^j \leq v < e^{j+1}} \frac{1}{v} \sum_{\substack{m,n=1 \\ mu \equiv nv \pmod{q} \\ mu \neq nv}}^N 1 \\
&\leq 2\varphi(q) \sum_{0 \leq i \leq j \leq J} e^{-i-j} \sum_{e^i \leq u < e^{i+1}} \sum_{e^j \leq v < e^{j+1}} \sum_{\substack{m,n=1 \\ mu \equiv nv \pmod{q} \\ mu \neq nv}}^N 1.
\end{aligned}$$

Therefore

$$M_2 \leq 2\varphi(q) \sum_{0 \leq i \leq j \leq J} e^{-i-j} T_{i,j}, \quad (4)$$

where  $T_{i,j}$  is the number of solutions  $(m, n, u, v)$  to the congruence

$$mu \equiv nv \pmod{q}, \quad 1 \leq m, n \leq N, \quad e^i \leq u < e^{i+1}, \quad e^j \leq v < e^{j+1},$$

with  $mu \neq nv$ .

If for a solution  $(m, n, u, v)$  we write  $mu = nv + kq$  with an integer  $k$  then we see that

$$1 \leq |k| \leq q^{-1} \max\{mu, nv\} \leq q^{-1} N \max\{e^{i+1}, e^{j+1}\} = e^{j+1} N/q.$$

Thus, there are  $O(e^j N/q)$  possible values for  $k$ . Clearly there are at most  $e^{i+1}$  possible values for  $u$  and  $N$  possible values  $m$ . Thus the product  $nv = mu - kq$  can take at most  $e^{i+j+2} N^2/q$  possible values and they are all of the size  $O(Nq^2) = O(q^3)$ . Therefore, we see from the bound on the divisor function (2) that when  $m, u$  and  $k$  are fixed then  $n$  and  $v$  can take at most  $q^{o(1)}$  possible values. Hence

$$T_{i,j} \leq e^{i+j} N^2 q^{-1+o(1)}$$

which after substitution in (4) gives

$$M_2 \leq J^2 \varphi(q) N^2 q^{-1+o(1)}$$

and the bound (3) follows. □

### 3 Final Remarks

As we have mentioned our result is nontrivial for  $q^\varepsilon \leq N \leq q^{1-\varepsilon}$ . However, the author sees no reason why an appropriate asymptotic formula cannot hold for even larger values of  $N$ , say up to  $q/2$ . It would be interesting to clarify this issue.

### References

- [1] G. H. Hardy and E. M. Wright, *An introduction to the theory of numbers*, Oxford Univ. Press, Oxford, 1979.
- [2] W. Zhang, ‘On the mean value of  $L$ -functions with the weight of character sums’, *J. Number Theory*, **128** (2008), 2459–2466.